Pioneer Journal of Mathematics and Mathematical Sciences
Volume 13, Number 1, 2015, Pages 15-37
This paper is available online on December 24, 2014 at http://www.pspchv.com/content_PJMMS.html

A NOTE ON $f^{\pm}$-ZAGREB INDICES IN RESPECT OF JACO GRAPHS, $J_n(1)$, $n \in \mathbb{N}$ AND THE INTRODUCTION OF KHAZAMULA IRREGULARITY

JOHAN KOK and VIVIAN MUKUNGUNUGWA

Tshwane Metropolitan Police Department
City of Tshwane
Republic of South Africa

Department of Mathematics and Applied Mathematics
University of Zimbabwe
City of Harare, Republic of Zimbabwe

Abstract

The topological graph indices $irr(G)$ related to the first Zagreb index, $M_1(G)$ and the second Zagreb index, $M_2(G)$ are of the oldest irregularity measures researched. Alberton [M. O. Albertson, The irregularity of a graph, Ars Combinatoria 46 (1997), 219-225] introduced the irregularity of G as

$$irr(G) = \sum_{e \in E(G)} imb(e), \quad imb(e) = |\, d(v) - d(u)\,|_{e=vu}.$$

Received October 1, 2014

2010 Mathematics Subject Classification: 05C07, 05C20, 05C38, 05C75, 05C85.

Keywords and phrases: total irregularity, irregularity, imbalance, Zagreb indices, $\pm$Fibonacci weight, total f-irregularity, Fibonaccian irregularity, $f^{\pm}$-Zagreb indices, Jaco graphs, Zeckendorf representation, Khazamula irregularity, Khazamula theorem.

The publishing of this paper was made possible through a publishing grant from Mogatladi Trading Enterprise CC, City of Tshwane, Republic of South Africa.

In the paper of Fath-Tabar [G. H. Fath-Tabar, Old and new Zagreb indices of graphs, MATCH Communications in Mathematical and in Computer Chemistry 65 (2011), 79-84], Alberton's indice was named the third Zagreb indice to conform with the terminology of chemical graph theory. Recently Ado et al. [H. Abdo and D. Dimitrov, The total irregularity of a graph, arXiv: 1207.5267v1 [math.CO], 24 July 2012] introduced the topological indice called total irregularity. The latter could be called the fourth Zagreb indice. We define the $\pm$ Fibonacci weight, $f_i^{\pm}$ of a vertex v_i to be $-f_{d(v_i)}$, if $d(v_i)$ is uneven and, $f_{d(v_i)}$, if $d(v_i)$ is even. From the aforesaid we define the $f^{\pm}$-Zagreb indices. This paper presents introductory results for the undirected underlying graphs of Jaco graphs, $J_n(1)$, $n \leq 12$. For more on Jaco graphs $J_n(1)$ see [J. Kok, P. Fisher, B. Wilkens, M. Mabula and V. Mukungunugwa, Characteristics of Finite Jaco Graphs, $J_n(1)$, $n \in \mathbb{N}$, arXiv: 1404.0484v1 [math.CO], 2 April 2014; J. Kok, P. Fisher, B. Wilkens, M. Mabula and V. Mukungunugwa, Characteristics of Jaco Graphs, $J_\infty(a)$, $a \in \mathbb{N}$, arXiv: 1404.1714v1 [math.CO], 7 April 2014]. Finally, we introduce the Khazamula irregularity as a new topological variant. We also present five open problems.

1. Introduction

The topological graph indices $irr(G)$ related to the first Zagreb index,

$$M_1(G) = \sum_{v \in V(G)} d^2(v) = \sum_{vu \in E(G)} (d(v) + d(u)),$$

and the second Zagreb index,

$$M_2(G) = \sum_{vu \in E(G)} d(v)d(u)$$

are of the oldest irregularity measures researched. Alberton [3] introduced the irregularity of G as

$$irr(G) = \sum_{e \in E(G)} imb(e), \quad imb(e) = |\, d(v) - d(u)\,|_{e=vu}.$$

In the paper of Fath-Tabar [7], Alberton's indice was named the third Zagreb indice to conform with the terminology of chemical graph theory. Recently Ado et al. [1]

introduced the topological indice called total irregularity and defined it,

$$irr_t(G) = \frac{1}{2} \sum_{u, v \in (G)} | d(u) - d(v) |.$$

The latter could be called the *fourth Zagreb indice*.

If the vertices of a simple undirected graph G on n vertices are labelled v_i, $i = 1, 2, 3, ..., n$, then the respective definitions may be

$$M_1(G) = \sum_{i=1}^{n} d^2(v_i) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} (d(v_i) + d(v_j))_{v_i u_j \in E(G)},$$

$$M_2(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} d(v_i) d(v_j)_{v_i u_j \in E(G)},$$

$$M_3(G) = \sum_{i=1}^{n-1} \sum_{j=2}^{n} | d(v_i) - d(v_j) |_{v_i u_j \in E(G)}$$

and

$$M_4(G) = irr_t(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} | d(v_i) - d(v_j) |$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} | d(v_i) - d(v_j) |$$

or

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} | d(v_i) - d(v_j) |.$$

For a simple graph on a singular vertex (1-empty graph), we define $M_1(G) = M_2(G) = M_3(G) = M_4(G) = 0$.

2. Zagreb Indices in Respect of $\pm$ Fibonacci Weights, $f^{\pm}$-Zagreb Indices

We define the $\pm$ Fibonacci weight, $f_i^{\pm}$ of a vertex v_i to be $-f_{d(v_i)}$, if $d(v_i) = i$ is uneven and, $f_{d(v_i)}$, if $d(v_i)$ is even. The $f^{\pm}$-Zagreb indices can now be defined as:

$$f^{\pm}Z_1(G) = \sum_{i=1}^{n}(f_i^{\pm})^2 = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(|\, f_i^{\pm}\,| + |\, f_j^{\pm}\,|)_{v_i u_j \in E(G)},$$

$$f^{\pm}Z_2(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(f_i^{\pm} \cdot f_j^{\pm})_{v_i u_j \in E(G)},$$

$$f^{\pm}Z_3(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\, f_i^{\pm} - f_j^{\pm}\,|_{v_i u_j \in E(G)}$$

and

$$f^{\pm}Z_4(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|\, f_i^{\pm} - f_j^{\pm}\,| = \sum_{i=1}^{n}\sum_{j=i+1}^{n}|\, f_i^{\pm} - f_j^{\pm}\,|$$

or

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|\, f_i^{\pm} - f_j^{\pm}\,|.$$

For a simple graph on a singular vertex (1-empty graph), we define

$$f^{\pm}Z_1(G) = f^{\pm}Z_2(G) = f^{\pm}Z_3(G) = f^{\pm}Z_4(G) = 0.$$

2.1. Application to Jaco graphs, $J_n(1)$, $n \in \mathbb{N}$

For ease of reference some definitions in [9] are repeated. A particular family of finite directed graphs (*order* 1) called Jaco Graphs and denoted by $J_n(1)$, $n \in \mathbb{N}$ are directed graphs derived from a particular well-defined infinite directed graph (*order* 1), called the 1-root digraph. The 1-root digraph has four fundamental properties which are; $V(J_\infty(1)) = \{v_i \mid i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc) then the tail is always a vertex v_i, $i < j$ and, if v_k, for the smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex k is $d(v_k) = k$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) v_t, $t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

Definition 2.1. The infinite Jaco graph $J_\infty(1)$ is defined by $V(J_\infty(1)) = \{v_i \mid i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j) \mid i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \geq j$, [9].

Definition 2.2. The family of finite Jaco graphs are defined by $\{J_n(1) \subseteq J_\infty(1) \mid n \in \mathbb{N}\}$. A member of the family is referred to as the Jaco graph, $J_n(1)$, [9].

Definition 2.3. The set of vertices attaining degree $\Delta(J_n(1))$ is called the *Jaconian vertices* of the Jaco graph $J_n(1)$, and denoted, $\mathbb{J}(J_n(1))$ or, $\mathbb{J}_n(1)$ for brevity, [9].

From [9] we have Bettina's theorem.

Theorem 2.1. *Let* $\mathbb{F} = \{f_0, f_1, f_2, f_3, ...\}$ *be the set of Fibonacci numbers and let* $n = f_{i_1} + f_{i_2} + \cdots + f_{i_r}$, $n \in \mathbb{N}$ *be the Zeckendorf representation of* n. *Then*

$$d^+(v_n) = f_{i_1 - 1} + f_{i_2 - 1} + \cdots + f_{i_r - 1}.$$

Note. The degree of vertex v_i, denoted $d(v_i)$ refers to the degree in $J_\infty(1)$ hence $d(v_i) = i$. In the finite Jaco graph the degree of vertex v_i is denoted $d(v_i)_{J_n(1)}$. The degree sequence is denoted

$$\mathbb{D}_n = (d(v_1)_{J_n(1)}, d(v_2)_{J_n(1)}, ..., d(v_n)_{J_n(1)}).$$

By convention $\mathbb{D}_{i+1} = \mathbb{D}_i \cup d(v_{i+1})_{J_n(1)}$.

2.1.1. Algorithm to determine the degree sequence of a finite Jaco graph, $J_n(1)$, $n \in \mathbb{N}$

Consider a finite Jaco Graph $J_n(1)$, $n \in \mathbb{N}$ and label the vertices $v_1, v_2, v_3, \ldots, v_n$.

Step 0. Set $n = n$. Let $i = j = 1$. If $j = n = 1$, let $\mathbb{D}_i = (0)$ and go to Step 6, else set $\mathbb{D}_i = \varnothing$ and go to Step 1.

Step 1. Determine the jth Zeckendorf representation say, $j = f_{i_1} + f_{i_2} + \cdots + f_{i_r}$ and go to Step 2.

Step 2. Calculate $d^+(v_j) = f_{i_1-1} + f_{i_2-1} + \cdots + f_{i_r-1}$, then go to Step 3.

Step 3. Calculate $d^-(v_j) = j - d^+(v_j)$, and let $d(v_j) = d^+(v_j) + d^-(v_j)$, then go to Step 4.

Step 4. If $d(v_j) \leq n$, set $d(v_j)_{J_n(1)} = d(v_j)$ else, set $d(v_j)_{J_n(1)} = d^-(v_j) + (n - j)$ and set $\mathbb{D}_j = \mathbb{D}_i \cup d(v_j)_{J_n(1)}$ and go to Step 5.

Step 5. If $j = n$ go to Step 6 else, set $i = i + 1$ and $j = i$ and go to Step 1.

Step 6. Exit.

2.1.2. Tabled values of $\mathbb{F}^{\pm}(J_n(1))$, for finite Jaco graphs, $J_n(1)$, $n \leq 12$

For illustration the adapted table below follows from the Fisher algorithm [9] for $J_n(1)$, $n \leq 12$. Note that the Fisher algorithm determines $d^+(v_i)$ on the assumption that the Jaco graph is always sufficiently large, so at least $J_n(1)$, $n \geq i + d^+(v_i)$. For a smaller graph the degree of vertex v_i is given by $d(v_i)_{J_n(1)} = d^-(v_i) + (n - i)$. In [9] Bettina's theorem describes an arguably, closed formula to determine $d^+(v_i)$. Since $d^-(v_i) = n - d^+(v_i)$ it is then easy to determine $d(v_i)_{J_n(1)}$ in a smaller graph $J_n(1)$, $n < i + d^+(v_i)$. The $f_i^{\pm}$ -sequence of $J_n(1)$ is denoted $\mathbb{F}^{\pm}(J_n(1))$.

Table 1

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i) = i - d^-(v_n)$	$\mathbb{F}^{\pm}(J_i(1))$
1	0	1	(0)
2	1	1	(–1, –1)
3	1	2	(–1, 1, –1)
4	1	3	(–1, 1, 1, –1)
5	2	3	(–1, 1, −2, 1, 1)
6	2	4	(–1, 1, −2, −2, −2, 1)
7	3	4	(–1, 1, −2, 3, 3, −2, −2)
8	3	5	(–1, 1, −2, 3, −5, 3, 3, −2)
9	3	6	(–1, 1, −2, 3, −5, −5, −5, 3, −2)
10	4	6	(–1, 1, −2, 3, −5, 8, 8, −5, 3, 3)
11	4	7	(–1, 1, −2, 3, −5, 8, –13, 8, −5, −5, 3)
12	4	8	(–1, 1, −2, 3, −5, 8, –13, –13, 8, 8, −5, 3)

It is known that a sequence $(d_1, d_2, d_3, ..., d_n)$ of non-negative integers is a degree sequence of some graph G if and only if $\sum_{i+i}^{n} d_i$ is even. It implies that a degree sequence has an even number of odd entries. Hence, we know that the $f_i^{\pm}$-sequence of $J_n(1)$ denoted, $\mathbb{F}^{\pm}(J_n(1))$, $n \in \mathbb{N}$ has an even number of, $-f_{d(v_i)}$ entries. Following from Table 1 the table below depicts the values $f^{\pm}Z_1(J_n(1))$, $f^{\pm}Z_2(J_n(1))$, $f^{\pm}Z_3(J_n(1))$ and $f^{\pm}Z_4(J_n(1))$ for $J_n(1)$, $n \leq 12$.

Table 2

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	$f^{\pm}Z_1(J_i(1))$	$f^{\pm}Z_2(J_i(1))$	$f^{\pm}Z_3(J_i(1))$	$f^{\pm}Z_4(J_i(1))$
1	0	1	0	0	0	0
2	1	1	2	1	0	0
3	1	2	3	–2	4	4
4	1	3	4	–1	4	8
5	2	3	8	–6	11	16
6	2	4	15	5	11	25
7	3	4	32	–26	35	56
8	3	5	62	–19	50	98
9	3	6	103	0	72	138
10	4	6	211	38	119	251
11	4	7	396	–238	210	402
12	4	8	604	–158	273	566

3. Khazamula Irregularity

Let $G^{\rightarrow}$ be a simple directed graph on $n \geq 2$ vertices labelled $v_1, v_2, v_3, \ldots, v_n$. Let all vertices v_i carry its $\pm$Fibonacci weight, $f_i^{\pm}$ related to $d(v_i) = d(v^+(v_i) + d^-(v_i))$. Also let vertex v_j be a head vertex of v_i and choose any $d(v_i^h) = \max(d(v_j)_{\forall v_j})$.

Definition 3.1. Let $G^{\rightarrow}$ be a simple directed graph on $n \geq 2$ vertices with each vertex carrying its $\pm$Fibonacci weight, $f_i^{\pm}$. For the function $f(x) = mx + c$, $x \in \mathbb{R}$ and $m, c \in \mathbb{Z}$, we define the Khazamula irregularity as:

$$irr_k(G^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|.$$

Note. Vertices v with $d^{+}(v) = 0,$ are headless and the corresponding integral terms to the summation are defined zero. Hence, $irr_k(K_1^{\rightarrow}) = 0.$

Let G be a simple connected undirected graph on n vertices which are labelled, $v_1, v_2, v_3, ..., v_n.$ Also let G have ε edges. It is known that G can be orientated in 2^{ε} ways, including the cases of isomorphism. Finding the relationship between the different values of $irr_k(G^{\rightarrow})$ and $irr_k^c(G^{\rightarrow})$ (to follow in Subsection 3.3) in respect of the different orientations for G in general is stated as an open problem. In this section, we give results in respect of particular orientations of paths, cycles, wheels and complete bipartite graphs.

3.1. irr_k for paths, cycles, wheels and complete bipartite graphs

Proposition 3.1. *For a directed path* $P_n^{\rightarrow},$ $n \geq 2$ *which is consecutively directed from left to right we have that the Khazamula irregularity,*

$$irr_k(P_n^{\rightarrow}) = \left| \frac{3}{2}(n-2)m + nc \right|.$$

Proof. Label the vertices of the directed path $P_n^{\rightarrow}$ consecutively from left to right $v_1, v_2, v_3, ..., v_n.$ From the definition

$$irr_k(P_n^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|,$$

it follows that we have

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right| = \left| \int_{-1}^{2} f(x)dx + \underbrace{\int_{1}^{2} f(x)dx + \cdots + \int_{1}^{2}}_{(n-3)\text{-}terms} + \int_{1}^{1} f(x)dx \right|.$$

So, we have

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right| = \left| \left(\frac{1}{2}mx^2 + cx\right)\Big|_{-1}^{2} + (n-3)\left(\frac{1}{2}mx^2 + cx\right)\Big|_{1}^{2} + 0 \right|$$

$$= \left| 2m + 2c - \frac{1}{2}m + c + (n-3)\left(2m + 2c - \frac{1}{2}m - c\right) \right|$$

$$= \left| \frac{3}{2}m + 3c + \frac{3}{2}(n-3)m + (n-3)c \right|$$

$$= \left| \frac{3}{2}(n-2)m + nc \right|. \qquad \square$$

Proposition 3.2. *For a directed cycle* $C_n^{\rightarrow}$ *which is consecutively directed clockwise we have that the Khazamula irregularity,*

$$irr_k(C_n^{\rightarrow}) = n\left| \frac{3}{2}m + c \right|.$$

Proof. Label the vertices of the directed cycle $C_n^{\rightarrow}$ consecutively clockwise $v_1, v_2, v_3, \ldots, v_n$. So vertices carry the $\pm$ Fibonacci weight, $f^{\pm}_{i_{\forall i}} = f_1 = 1$. Also a head vertex is always unique with degree $= 2$. From the definition

$$irr_k(C_n^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|,$$

it follows that we have

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right| = \left| \underbrace{\int_1^2 f(x)dx + \int_1^2 f(x)dx + \cdots + \int_1^2 f(x)dx}_{n\text{-}terms} \right|$$

$$= \left| n\left(\frac{1}{2}mx^2 + cx\right)\Big|_1^2 \right|$$

$$= \left| n\left(2m + 2c - \frac{1}{2}m - c\right) \right|$$

$$= \left| n\left(\frac{3}{2}m + c\right) \right| = n\left| \frac{3}{2}m + c \right|. \qquad \square$$

Proposition 3.3. *For a directed Wheel graph* $W^{\rightarrow}_{(1,n)}$ *with the axle vertex* u_1 *and the wheel vertices* $v_1, v_2, \ldots, v_n$ *and the spokes directed* $(u_1, v_i)_{\forall i}$ *and the wheel* *vertices directed consecutively clockwise* $v_1, v_2, \ldots, v_n$, *we have that*

$$irr_k(W_{(1,n)}^{\rightarrow})\begin{cases} = \left| \dfrac{\left(5n - f_n^2 + 9\right)}{2} m + (5n - f_n + 3)c \right|, & \textit{if } n \textit{ is even,} \\ = \left| \dfrac{\left(5n - f_n^2 + 9\right)}{2} m + (5n + f_n + 3)c \right|, & \textit{if } n \textit{ is uneven.} \end{cases}$$

Proof. Consider a Wheel graph $W_{(1,n)}^{\rightarrow}$ with the axle vertex u_1 and the wheel vertices $v_1, v_2, \ldots, v_n$ and the spokes directed $(u_1, v_i)_{\forall i}$ and the wheel vertices directed consecutively clockwise $v_1, v_2, \ldots, v_n$.

Case 1. If *n* is even, then $d(u_1)$ is even and carries the $\pm$ Fibonacci weight, f_n. Obviously the wheel vertices have $d(v_i) = 3_{\forall i}$, hence carry the $\pm$ Fibonacci weight, $f_3 = -2_{\forall v_i}$. So from the definition of the Khazamula irregularity we have that:

$$irr_k(W_{(1,n)}^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|$$

$$= \left| n\int_{-2}^{3} f(x)dx + \int_{f_n}^{3} f(x)dx \right|$$

if *n* is even. This results in,

$$irr_k = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|$$

$$= \left| n\left(\frac{9}{2}m + 3c - 2m + 2c\right) + \left(\frac{9}{2}m + 3c - \frac{f_n^2}{2}m - f_n c\right) \right|$$

$$= \left| \frac{5}{2}nm + 5nc + \frac{9}{2}m + 3c - \frac{f_n^2}{2}m - f_n c \right|$$

$$= \left| \frac{\left(5n - f_n^2 + 9\right)}{2} m + (5n - f_n + 3)c \right|.$$

Case 2. If n is uneven, then $d(u_1)$ is uneven and carries the $\pm$ Fibonacci weight, $-f_n$. So in the Riemann integral $\int_{-f_n}^{3} f(x)dx$ we have $\left(\frac{9}{2}m + 3c - \frac{f_n^2}{2}m + f_n c\right)$. So the result

$$irr_k(W_{(1,n)}^{\rightarrow}) = \left| \frac{\left(5n - f_n^2 + 9\right)}{2} m + (5n + f_n + 3)c \right|$$

if n is uneven, follows. □

Consider the complete bipartite graph $K_{(n,m)}$ and call the n vertices the left-side vertices and the m vertices the right-side vertices. Orientate $K_{(n,m)}$ strictly from left-side vertices to right-side vertices to obtain $K_{(n,m)}^{l\rightarrow r}$.

Proposition 3.4. *For the directed graph* $K_{(n,m)}^{l\rightarrow r}$, *we have that*

$$irr_k(K_{(n,m)}^{l\rightarrow r}) \begin{cases} = \left| \frac{\left(n^3 - nf_m^2\right)}{2} m + (n^2 - nf_m)c \right|, & \text{if } m \text{ is even,} \\ = \left| \frac{\left(n^3 - nf_m^2\right)}{2} m + (n^2 + nf_m)c \right|, & \text{if } m \text{ is uneven.} \end{cases}$$

Proof. For the directed graph $K_{(n,m)}^{l\rightarrow r}$ we have that all left-side vertices say $v_1, v_2, \ldots, v_n$ have $d^+(v_i) = m$, whilst all right-side vertices say $u_1, u_2, \ldots, u_m$ have $d^-(u_i) = n$ and $d^+(u_i) = 0$.

Case 1. If m is even it follows from the definition that,

$$irr_k(K_{(n,m)}^{l\to r}) = n\left|\int_{f_m}^{n} f(x)dx\right|.$$

So, we have that

$$irr_k(K_{(n,m)}^{l\to r}) = n\left|\left(\frac{1}{2}mx^2 + cx\right)|_{f_m}^{n}\right|$$

$$= n\left|\frac{n^2}{2}m + nc - \left(\frac{f_m^2}{2}m + f_m c\right)\right|$$

$$= \left|\frac{(n^3 - nf_m^2)}{2}m + (n^2 - nf_m)c\right|.$$

Case 2. If m is uneven the left-side vertices all carry the $\pm$Fibonacci weight, $-f_m$. Hence, the result follows as in Case 1, accounting for $-f_m$. □

Example problem 1. Let $n = 1$ or 5 and $f(x) = mx$. Prove that $irr_k(K_{(1,n)}^{\to}) =$ 0 or $|12m|$ and,

$$irr_k(K_{(1,n)}^{l\to r})\begin{cases} = 0 \\ or \\ = 5(irr_k(K_{(1,n)}^{\to})) = 60|m|. \end{cases}$$

Proof. Let $n = 1$ and let $f(x) = mx$. From the definition of $irr_k(G^{\to})$ it follows that

$$irr_k(K_{(1,n)}^{\to}) = \int_{-1}^{1}|mx \cdot dx|_{for-v_1} = \left|\frac{1}{2}mx^2|_{-1}^{1}\right| = 0.$$

We also have that

$$irr_k(K_{(1,n)}^{\to}) = \int_{-1}^{1}|mx \cdot dx|_{for-u_1}$$

$$= \left| \frac{1}{2} m x^2 \big|_{-1}^{1} \right| = 0.$$

Let $n = 5$ and let $f(x) = mx$. Now, we have that

$$irr_k(K^{\rightarrow}_{(1,n)}) = \int_{-5}^{1} | mx \cdot dx |_{for-v_1} = \left| \frac{1}{2} m x^2 \big|_{-5}^{1} \right| = | 12m |.$$

For $irr_k(K^{\rightarrow}_{(1,n)})$ we have

$$\sum_{i=1}^{5} \int_{-1}^{5} | mx \cdot dx |_{for-u_i, i=1, 2, \dots, 5}$$

$$= 5\left(\int_{-1}^{5} | mx \cdot dx | \right) = 5\left| \frac{1}{2} m x^2 \big|_{-1}^{5} \right| = 5| 12m | = 60| m |.$$

3.2. Khazamula's theorem

Consider two simple connected directed graphs, $G^{\rightarrow}$ and $H^{\rightarrow}$. Let the vertices of $G^{\rightarrow}$ be labelled $v_1, v_2, \dots, v_n$ and the vertices of $H^{\rightarrow}$ be labelled $u_1, u_2, \dots, u_m$. Define the directed join as $(G^{\rightarrow} + H^{\rightarrow})^{\rightarrow}$ conventionally, with the arcs $\{(v_i, u_j) | \forall v_i \in V(G^{\rightarrow}), u_j \in V(H^{\rightarrow})\}$.

Theorem 3.5. *Consider two simple connected directed graphs, $G^{\rightarrow}$ on n vertices and $H^{\rightarrow}$ on m vertices then,*

$$irr_k((G^{\rightarrow} + H^{\rightarrow})^{\rightarrow}) = \left| n \int_{f_i^{\pm} |_{v_i \in V((G^{\rightarrow} + H^{\rightarrow})^{\rightarrow})}}^{\Delta(H^{\rightarrow})+n} f(x)\, dx + \sum_{i=1}^{m} \int_{f_{d(u_i)+1}}^{d(u_i^h)+1} f(x) dx \right|.$$

Proof. Note that in the graph $G^{\rightarrow}$ the maximum degree

$$\Delta(G^{\rightarrow}) = \max((d^{+}(v_\ell) + d^{-}(v_\ell)) \leq n - 1$$

for at least one vertex v_ℓ. If such a vertex v_ℓ is indeed the head vertex of a vertex v_t, then

$$\sum_{i=1}^{n}\left|\int_{f_i^{\pm}}^{d(v_i^h)} f(x)\,dx\right|,$$

will contain the term $\int_{f_i^{\pm}}^{\Delta(G)} f(x)\,dx.$

In $H^{\rightarrow}$ the maximum degree $\Delta(H^{\rightarrow}) = \max(d^{+}(u_s) + d^{-}(u_s)) \geq 1$ for some vertex u_s. Hence, in the directed graph $(G^{\rightarrow} + H^{\rightarrow})^{\rightarrow}$, all terms of

$$\sum_{i=1}^{n}\left|\int_{f_i^{\pm}}^{d(v_i^h)} f(x)\,dx\right|$$

reduces to zero and are replaced by the terms

$$\int_{f_i^{\pm}|_{v_i \in V((G^{\rightarrow}+H^{\rightarrow})^{\rightarrow})}}^{\Delta(H^{\rightarrow})+n} f(x)\,dx,$$

because $\Delta(G^{\rightarrow}) \leq n-1 < \Delta(H^{\rightarrow}) + n.$

In respect of $H^{\rightarrow}$ we have that each $d(u_i)_{\forall i}$ increases by exactly 1 so the value of $f_{d(u_i)_{\forall i}}$ switches between $\pm$ and adopts the value $f_{d(u_i)+1}$. Similarly all *head vertices*' degree increases by exactly 1. These observations result in

$$irr_k((G^{\rightarrow} + H^{\rightarrow})^{\rightarrow}) = \left| n\int_{f_i^{\pm}|_{v_i \in V((G^{\rightarrow}+H^{\rightarrow})^{\rightarrow})}}^{\Delta(H^{\rightarrow})+n} f(x)\,dx + \sum_{i=1}^{m}\int_{f_{d(u_i)+1}}^{d(u_i^h)+1} f(x)\,dx \right|. \quad \square$$

Example problem 2. An application of the Khazamula theorem to the graph $(C_n^{\rightarrow} + K_1)^{\rightarrow}$ in respect of $f(x) = mx$, results in

$$irr_k((C_n^{\rightarrow} + K_1)^{\rightarrow}) = \frac{1}{3}(n^2 - 4)\,irr_k(C_n^{\rightarrow})|_{f(x)=mx}\ .$$

3.3. Khazamula *c*-irregularity for orientated paths, cycles, wheels and complete bipartite graphs

Let $f(x) = \sqrt{r^2 - x^2}$, $x \in \mathbb{R}$ and $r = \max\{d(v_i)_{\forall v_i, d^{-}(v_i) \geq 1}$, or $|\,(f_i^{\pm})\,|_{\forall v_i}\}$.

We define Khazamula c-irregularity as

$$irr_k^c(G^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|.$$

It is known that

$$\int_a^b \sqrt{r^2 - x^2}\, dx = \left(\frac{1}{2} x\sqrt{r^2 - x^2} + \frac{r^2}{2} \arcsin \frac{x}{r} \right)\Big|_a^b.$$

Also note that arcsinθ applies to $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ to ensure a singular value for the respective integral terms.

Proposition 3.6. *For a directed path $P_n^{\rightarrow}$, $n \geq 3$ which is consecutively directed from left to right we have that the Khazamula c-irregularity,*

$$irr_k^c(P_n^{\rightarrow}) = (n-2)\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right).$$

Proof. Label the vertices of the directed path $P_n^{\rightarrow}$, $n \geq 3$ consecutively from left to right $v_1, v_2, v_3, \ldots, v_n$. Note that $r = \max\{d(v_i)_{\forall v_i}, \text{ or } |(f_i^{\pm})|_{\forall v_i}\} = 2.$ From the definition

$$irr_k^c(P_n^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|,$$

it follows that we have

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right| = \left| \int_{-1}^{2} f(x)dx + \underbrace{\int_1^2 f(x)dx + \cdots + \int_1^2}_{(n-3)\text{-}terms} + \int_1^1 f(x)dx \right|$$

So, we have

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|$$

$$= \left| \left(\frac{1}{2} x\sqrt{r^2 - x^2} + \frac{r^2}{2} \arcsin \frac{x}{r} \right)\Big|_{-1}^2 + (n-3)\left(\frac{1}{2} x\sqrt{r^2 - x^2} + \frac{r^2}{2} \arcsin \frac{x}{r} \right)\Big|_1^2 \right|$$

$$= \left| \left(\frac{1}{2} x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2} \right)\Big|_{-1}^{2} + (n-3)\left(\frac{1}{2} x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2} \right)\Big|_{1}^{2} \right|$$

$$= \left| \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) + (n-3)\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right|$$

$$= \left| (n-2)\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right|$$

$$= (n-2)\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right). \qquad \square$$

Proposition 3.7. *For a directed cycle* $C_n^{\rightarrow}$ *which is consecutively directed clockwise we have that the Khazamula c-irregularity,* $irr_k^c(C_n^{\rightarrow}) = n\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$.

Proof. Label the vertices of the directed cycle $C_n^{\rightarrow}$ consecutively clockwise $v_1, v_2, v_3, \ldots, v_n$. So all vertices carry the $\pm$ Fibonacci weight, $f_{i_{\forall i}}^{\pm} = f_1 = 1$. Also a head vertex is always unique with degree $= 2$. So

$$r = \max\{d(v_i)_{\forall v_i}, \text{ or } |(f_i^{\pm})|_{\forall v_i}\} = 2.$$

From the definition

$$irr_k(C_n^{\rightarrow}) = \sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|,$$

it follows that we have

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right| = \left| \underbrace{\int_1^2 f(x)dx + \int_1^2 f(x)dx + \cdots + \int_1^2 f(x)dx}_{n\text{-terms}} \right|$$

$$= n\left| \left(\frac{1}{2} x\sqrt{4 - x^2} + 2\arcsin\frac{x}{2} \right)\Big|_{1}^{2} \right|$$

$$= n\left| \left(0 + 2\arcsin 1 - \frac{\sqrt{3}}{2} - 2\arcsin\frac{1}{2} \right) \right|$$

$$= n\left|\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right|$$

$$= n\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right). \qquad \square$$

Proposition 3.8. *For a directed Wheel graph* $W_{(1,n)}^{\rightarrow}$ *with the axle vertex* u_1 *and the wheel vertices* $v_1, v_2, \ldots, v_n$ *and the spokes directed* $(u_1, v_i)_{\forall i}$ *and the wheel vertices directed consecutively clockwise* $v_1, v_2, \ldots, v_n$, *we have that*:

$$irr_k W_{(1,n)}^{\rightarrow} \begin{cases} = 4\sqrt{5} + 9\pi + 18\arcsin\frac{2}{3}, & \text{if } n = 3 \text{ or } 4, \\ = \left|\frac{3}{2}(n+1)\sqrt{f_n^2 - 9} + (n+1)\frac{f_n^2}{2}\arcsin\frac{3}{f_n} - \frac{f_n^2\pi}{4} + A\right|, & \text{if } n \geq 6 \text{ and even}, \\ = \left|\frac{3}{2}(n+1)\sqrt{f_n^2 - 9} + (n+1)\frac{f_n^2}{2}\arcsin\frac{3}{f_n} + \frac{f_n^2\pi}{4} + B\right|, & \text{if } n \geq 5 \text{ and uneven}, \end{cases}$$

with

$$A = n\left(\sqrt{f_n^2 - 4} + \frac{f_n^2}{2}\arcsin\frac{2}{f_n}\right) \text{ and } B = n\left(\sqrt{f_n^2 - 4} - \frac{f_n^2}{2}\arcsin\frac{2}{f_n}\right).$$

Proof. Consider a Wheel graph $W_{(1,n)}^{\rightarrow}$ with the axle vertex u_1 and the wheel vertices $v_1, v_2, \ldots, v_n$ and the spokes directed $(u_1, v_i)_{\forall i}$ and the wheel vertices directed consecutively clockwise $v_1, v_2, \ldots, v_n$.

Case 1. If $n = 3$, then we have that

$$irr_k^c(W_{(1,3)}^{\rightarrow}) = \left|\underbrace{\int_{-2}^{3}\sqrt{9 - x^2}\,dx}_{for,\, u_1} + 3\underbrace{\int_{-2}^{3}\sqrt{9 - x^2}\,dx}_{for,\, v_i}\right|, \quad i = 1, 2, 3.$$

Therefore,

$$irr_k^c(W_{(1,3)}^{\rightarrow}) = 4\left|\left(\int_{-2}^{3}\sqrt{9 - x^2}\,dx\right)\right|$$

$$= 4\left|\left(\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\arcsin\frac{x}{3}\right)\Big|_{-2}^{3}\right|$$

$$= 4\left|\left(\frac{9}{2}\arcsin 1 - \left(-\sqrt{9-4} - \frac{9}{2}\arcsin\frac{2}{3}\right)\right)\right|$$

$$= 4\sqrt{5} + 9\pi + 18\arcsin\frac{2}{3}.$$

If $n = 4$, then

$$irr_k^c(W_{\overrightarrow{(1,4)}}) = \left| \underbrace{\int_3^3 \sqrt{9-x^2}\,dx}_{for,\, u_1} + 4\underbrace{\int_{-2}^3 \sqrt{9-x^2}\,dx}_{for,\, v_i} \right|, \quad i = 1, 2, 3, 4.$$

Hence, the result follows.

Case 2. If $n \geq 6$ and even, then we have

$$irr_k^c(W_{\overrightarrow{(1,n)}}) = \left| \underbrace{\int_{f_n}^3 \sqrt{f_n^2 - x^2}\,dx}_{for,\, u_1} + n\underbrace{\int_{-2}^3 \sqrt{f_n^2 - x^2}\,dx}_{for,\, v_i} \right|, \quad i = 1, 2, \ldots, n.$$

So, we have

$$irr_k^c(W_{\overrightarrow{(1,n)}})$$

$$= \left|\left(\frac{1}{2}x\sqrt{f_n^2 - x^2} + \frac{f_n^2}{2}\arcsin\frac{x}{f_n}\right)\Big|_{f_n}^{3} + n\left(\frac{1}{2}x\sqrt{f_n^2 - x^2} + \frac{f_n^2}{2}\arcsin\frac{x}{f_n}\right)\Big|_{-2}^{3}\right|$$

$$= \left|\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} - \left(\frac{f_n}{2}\sqrt{f_n^2 - f_n^2} + \frac{f_n^2}{2}\arcsin 1\right)\right.$$

$$\left. + n\left(\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} - \left(-\sqrt{f_n^2 - 4} - \frac{f_n^2}{2}\arcsin\frac{2}{f_n}\right)\right)\right|$$

$$= \left|\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} - \left(\frac{f_n}{2}\sqrt{f_n^2 - f_n^2} + \frac{f_n^2}{2}\arcsin 1\right)\right.$$

$$+ n\left(\frac{3}{2}\sqrt{f_n^2 - 9} + \frac{f_n^2}{2}\arcsin\frac{3}{f_n} + \sqrt{f_n^2 - 4} + \frac{f_n^2}{2}\arcsin\frac{2}{f_n}\right)\Bigg|$$

$$= \left|\frac{3}{2}(n+1)\sqrt{f_n^2 - 9} + (n+1)\frac{f_n^2}{2}\arcsin\frac{3}{f_n} + \frac{f_n^2\pi}{4} + A\right|,$$

with $A = n\left(\sqrt{f_n^2 - 4} + \frac{f_n^2}{2}\arcsin\frac{2}{f_n}\right)$.

Case 3. Similar to Case 2 and accounting for $n \geq 5$ and uneven. □

Consider the complete bipartite graph $K_{(n,m)}$ and call the n vertices the left-side vertices and the m vertices the right-side vertices. Orientate $K_{(n,m)}$ strictly from left-side vertices to right-side vertices to obtain $K_{(n,m)}^{l\to r}$.

Proposition 3.9. *For the directed graph* $K_{(n,m)}^{l\to r}$ *we have that*:

$$irr_k^c(K_{(n,m)}^{l\to r})\begin{cases} = \left|\frac{n^2\pi}{4} - A\right|, & \text{if } n \geq f_m \text{ and } m \text{ is even},\\ = \left|\frac{n^2\pi}{4} + A\right|, & \text{if } n \geq f_m \text{ and } m \text{ is uneven},\\ = \left|B - \frac{f_m^2\pi}{4}\right|, & \text{if } f_m > n \text{ and } m \text{ is even},\\ = \left|B + \frac{f_m^2\pi}{4}\right|, & \text{if } f_m > n \text{ and } m \text{ is uneven},\end{cases}$$

with $A = \frac{f_m}{2}\sqrt{n^2 - f_m^2} + \frac{n^2}{2}\arcsin\frac{f_m}{n}$ *and* $B = \frac{n}{2}\sqrt{f_m^2 - n^2} + \frac{f_m^2}{2}\arcsin\frac{n}{f_m}$.

Proof. For the directed graph $K_{(n,m)}^{l\to r}$ we have that all left-side vertices say $v_1, v_2, \ldots, v_n$ have $d^+(v_i) = m$, whilst all right-side vertices say $u_1, u_2, \ldots, u_m$ have $d^-(u_i) = n$ and $d^+(u_i) = 0$.

Case 1. Since $d^{+}(u_i) = 0, \ \forall i$ the terms in

$$\sum_{i=1}^{n} \left| \int_{f_i^{\pm}}^{d(v_i^h)} f(x)dx \right|,$$

stem from vertices $v_i, \ \forall i$ only. Furthermore, since

$$r = \max\{d(u_i)_{\forall i, d^{-}(u_i) \geq 1}, \text{ or } f_m\}$$

and $n \geq f_m$, we have $r = n$.

It follows that

$$irr_k^c(K_{(n,m)}^{l \to r}) = n \left| \int_{f_m}^{n} \sqrt{n^2 - x^2} dx \right|$$

$$= \left| \left(\frac{1}{2} x\sqrt{n^2 - x^2} + \frac{n^2}{2} \arcsin \frac{x}{n} \right) \Big|_{f_m}^{n} \right|$$

$$= \left| \frac{n^2}{2} \arcsin 1 - \left(\frac{f_m}{2} \sqrt{n^2 - f_m^2} + \frac{n^2}{2} \arcsin \frac{f_m}{n} \right) \right|$$

$$= \left| \frac{n^2 \pi}{4} - A \right|,$$

with

$$A = \frac{f_m}{2} \sqrt{n^2 - f_m^2} + \frac{n^2}{2} \arcsin \frac{f_m}{n}.$$

Case 2. Similar to Case 1 and accounting for m is uneven.

Case 3. Similar to Case 1 and accounting for $f_m > n, \ m$ is even.

Case 4. Similar to Case 1 and accounting for $f_m > n, \ m$ is uneven. □

[**Open problem:** If possible, generalize Khazamula's irregularity for simple directed graphs.]

[**Open problem:** Find a closed or, recursive formula for $f^{\pm}Z_1(J_n(1))$, $f^{\pm}Z_2(J_n(1))$, $f^{\pm}Z_3(J_n(1))$ and $f^{\pm}Z_4(J_n(1))$.]

[**Open problem:** Where possible, describe the terms of the Khazamula theorem in terms of $irr_k(G^{\rightarrow})$ and $irr_k(H^{\rightarrow})$ for specialised classes of simple directed graphs.]

[**Open problem:** If possible, formulate and prove Khazamula's *c*-Theorem related to Khazamula *c*-irregularity for simple directed graphs in general.]

[**Open problem:** Let G be a simple connected undirected graph on n vertices labelled, $v_1, v_2, v_3, \ldots, v_n$. Also let G have ε edges. It is known that G can be orientated in 2^{ε} ways, including the cases of isomorphism. Find the relationship between the different values of $irr_k(G^{\rightarrow})$ in respect of the different orientations.]

[**Open problem:** Let G be a simple connected undirected graph on n vertices labelled, $v_1, v_2, v_3, \ldots, v_n$. Also let G have ε edges. It is known that G can be orientated in 2^{ε} ways, including the cases of isomorphism. Find the relationship between the different values of $irr_k^c(G^{\rightarrow})$ in respect of the different orientations.]

References

[1] H. Abdo and D. Dimitrov, The total irregularity of a graph, arXiv: 1207.5267v1 [math.CO], 24 July 2012.

[2] Y. Alavi, A. Boals, G. Chartrand, P. Erdös, R. Graham and O. Oellerman, *k*-path irregular graphs, Congressus Numerantium 65 (1988), 201-210.

[3] M. O. Albertson, The irregularity of a graph, Ars Combinatoria 46 (1997), 219-225.

[4] A. R. Ashrafi, T. Došlić and A. Hamzeha, The Zagreb coindices of graph operations, Discrete Applied Mathematics 158 (2010), 1571-1578.

[5] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan Press, London, 1976.

[6] D. Cvetković and P. Rowlinson, On connected graphs with maximal index, Publications de I'Institut Mathematique Beograd 44 (1988), 29-34.

[7] G. H. Fath-Tabar, Old and new Zagreb indices of graphs, MATCH Communications in Mathematical and in Computer Chemistry 65 (2011), 79-84.

[8] M. A. Henning and D. Rautenbach, On the irregularity of bipartite graphs, Discrete Mathematics 307 (2007), 1467-1472.

[9] J. Kok, P. Fisher, B. Wilkens, M. Mabula and V. Mukungunugwa, Characteristics of Finite Jaco Graphs, $J_n(1)$, $n \in \mathbb{N}$, arXiv: 1404.0484v1 [math.CO], 2 April 2014.

[10] J. Kok, P. Fisher, B. Wilkens, M. Mabula and V. Mukungunugwa, Characteristics of Jaco Graphs, $J_\infty(a)$, $a \in \mathbb{N}$, arXiv: 1404.1714v1 [math.CO], 7 April 2014.

[11] J. Kok, Total irregularity and f_t-irregularity of Jaco graphs, $J_n(1)$, $n \in \mathbb{N}$, arXiv: 1406.6168v1 [math.CO], 24 June 2014.

[12] J. Kok, Total irregularity of graphs resulting from edge-joint, edge-transformation and $irr_t^+(G)$ and $irr_t^-(G)$ of directed graphs, arXiv: 1406.6863v2 [math.CO], 26 June 2014.

[13] Y. Zhu, L. You and J. Yang, The minimal total irregularity of graphs, arXiv: 1404.0931v1 [math.CO], 3 April 2014.